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Renormalisation of random walks with memory

S É Derkachov†, J Honkonen‡ and A N Vasil'ev†

† Department of Theoretical Physics, State University of Leningrad, Ul'yanovskaya 1, Staryi Petergof, 198904 Leningrad, USSR

‡ Research Institute for Theoretical Physics, University of Helsinki, Siltavuorenpenger 20 C, SF-00170 Helsinki, Finland

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Abstract. The renormalisation of field-theoretic models of the 'true' self-avoiding random walk is analysed. For short-range interactions, the field theory is shown to be renormalisable only in a special case, which corresponds to the problem of a random walk in a random environment, and non-renormalisable otherwise. The long-range version of the field theory is shown to be renormalisable in a larger region of the parameter space. In the case which corresponds to the original long-range 'true' self-avoiding random walk, the renormalisation group analysis is carried out to two-loop order. Anomalous dimensions of various composite operators are calculated and the crossover between scaling regimes described by short-range and long-range models is discussed.

1. Introduction

Models of stochastic processes incorporating effects of long memory have attracted considerable interest [1-6]. An early model of this kind is the 'true' self-avoiding random walk (TSAW) [1], in which the random walk dynamically tends to avoid places it has visited earlier. This model has statistical properties which drastically differ from those of the usual self-avoiding walk problem related to the account of excluded-volume effects in polymer statistics [7]. It was soon realised that the TSAW could be cast into the form of a field theory [3], and thus field-theoretic methods [8] could be used to study the asymptotic behaviour of such walks. In the original TSAW, the self-interaction of the random walk was local in space, but the generalisation to long-range repulsion has also been suggested and analysed in a similar fashion [4].

In this paper we show that in earlier work an infinite set of marginal operators has been overlooked, the account of which renders the field theory, in general, non-renormalisable. More explicitly, the field-theoretic model [3] is characterised by three coupling constants g_1, g_2 and g_3, and we show that, in the space of these three parameters, the model is renormalisable only on the line $g_1 = g_3$, $g_2 = 0$, where its asymptotic behaviour coincides with that of the model of random walk in random velocity field [9]. This implies that the model has no predictive power at and below its upper critical dimension $d_c = 2$. The long-range version of this model [4] also turns out to be non-renormalisable for arbitrary coupling constants. However, it is renormalisable in the plane $g_2 = 0$, and we have calculated to two-loop order the beta function and the anomalous dimension of the diffusion coefficient of the long-range TSAW (to which corresponds the line $g_2 = g_3 = 0$). We have also calculated, to one-loop order, anomalous dimensions of several composite operators relevant to the analysis

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of the crosssover from the scaling regime described by the long-range model to that described by the short-range model, which takes place in the limit $\alpha \to 0$ (α is the exponent, which characterises the power-like behaviour of the interaction in the long-range model). We also show that on the line $g_1 = g_2 = 0$ the anomalous dimension of the diffusion coefficient is determined exactly by the fixed-point equation of the RG.

The paper is set up as follows. In section 2 we present a novel derivation of the field-theoretic formalism from the original stochastic equations [3, 4], without resort to the Fock-space formalism for classical objects [10]. Section 3 is devoted to the analysis of the renormalisability of the corresponding field theory in both short-range and long-range cases. The results of the standard renormalisation group (RG) analysis are also exposed here. In section 4 we calculate the anomalous dimensions of various composite operators of the long-range TSAW, and discuss the behaviour of the model at small α. In section 5 we summarise the main results of the paper. Part of the results of this paper has been presented earlier [5].

2. Derivation of the field theory

In the continuum limit, the TSAW is defined by the equations [3]

$$\frac{\mathrm{d}\boldsymbol{R}(t)}{\mathrm{d}t} = -g_1 \nabla \rho(\boldsymbol{R}(t), t) + \eta(t) \tag{1}$$

$$\frac{\partial \rho(x, t)}{\partial t} = \delta(x - \boldsymbol{R}(t)) \tag{2}$$

where $\boldsymbol{R}$ is the position of the random walker, and η is a Gaussian noise with zero mean and correlation function $\overline{\eta_m(t)\eta_n(t')} = 2D_0\delta(t'-t)\delta_{mn}$, where D_0 is the 'bare' diffusion coefficient. These equations describe the short-range TSAW, whereas for the long-range version (1) is replaced by [4]

$$\frac{\mathrm{d}\boldsymbol{R}(t)}{\mathrm{d}t} = -g_1 \nabla \phi(\boldsymbol{R}(t), t) + \eta(t) \tag{3}$$

where the function ϕ is related to ρ as

$$\phi(\boldsymbol{x}, t) = \int \mathrm{d}\boldsymbol{x}'\, K(\boldsymbol{x} - \boldsymbol{x}')\rho(\boldsymbol{x}', t)$$

with the kernel $K(\boldsymbol{x})$ defined by

$$K(\boldsymbol{x}) = \int \frac{\mathrm{d}\boldsymbol{q}}{(2\pi)^d} \frac{\exp(\mathrm{i}\boldsymbol{qx})}{\boldsymbol{q}^{2\alpha}}. \tag{4}$$

We will be interested in the probability distribution $P(\boldsymbol{x}, t)$ of the random walks started at the origin

$$P(\boldsymbol{x}, t) = \overline{\delta(\boldsymbol{x} - \boldsymbol{R}(t))}$$

where $\boldsymbol{R}(t)$ is the solution of the problem (1), (2) with the initial condition $\boldsymbol{R}(0) = 0$. For the retarded Green function

$$G(\boldsymbol{x}, t) = \theta(t)P(\boldsymbol{x}, t)$$

a functional integral representation has been constructed using sophisticated Fock-space treatment of classical objects [10]. We shall present here an alternative derivation of the field theory, which is fairly straightforward, and conceptually simpler than the Fock space method.

To this end, we introduce the generalised distribution function $\tilde{P}$ defined by

$$\tilde{P}(\boldsymbol{x}, t; \boldsymbol{A}) \equiv \overline{\delta(x-\boldsymbol{R}(t)) \exp\left(\int_0^t \mathrm{d}\tau\, \boldsymbol{A}(\tau)\boldsymbol{R}(\tau)\right)} \tag{5}$$

and depending functionally on an auxiliary vector function $\boldsymbol{A}$. Obviously $P(\boldsymbol{x}, t) = \tilde{P}(\boldsymbol{x}, t; 0)$. Differentiating the definition (5) with respect to time t we obtain

$$\begin{aligned}\partial_t \tilde{P}(\boldsymbol{x}, t; \boldsymbol{A}) &= \overline{\delta(\boldsymbol{x}-\boldsymbol{R}(t))\boldsymbol{A}(t)\boldsymbol{R}(t) \exp\left(\int \mathrm{d}\tau\, \boldsymbol{AR}\right)} \\ &\quad \overline{-\nabla\delta(\boldsymbol{x}-\boldsymbol{R}(t))\frac{\mathrm{d}\boldsymbol{R}(t)}{\mathrm{d}t}\exp\left(\int \mathrm{d}\tau\, \boldsymbol{AR}\right)} \\ &= \boldsymbol{A}(t)\boldsymbol{x}\tilde{P}(\boldsymbol{x}, t; \boldsymbol{A}) \\ &\quad \overline{+\nabla\delta(\boldsymbol{x}-\boldsymbol{R}(t))\left(g_1\int_0^t \mathrm{d}\tau\, \nabla K(\boldsymbol{R}(t)-\boldsymbol{R}(\tau)) - \eta(t)\right)\exp\left(\int \mathrm{d}\tau\, \boldsymbol{AR}\right)}\end{aligned}$$

where the stochastic equation (3) has been used. The noise η has a Gaussian distribution, therefore the identity

$$\overline{\eta(t)F(\eta)} = 2D_0 \overline{\frac{\delta F(\eta)}{\delta\eta(t)}}$$

holds for an arbitrary functional F of η. Using this identity, and taking also into account that $\delta R_n(t)/\delta\eta_m(t) = \delta_{nm}/2$, we arrive at the equation

$$\begin{aligned}\partial_t \tilde{P}(\boldsymbol{x}, t; \boldsymbol{A}) &= (D_0\nabla^2 + \boldsymbol{Ax})\tilde{P}(\boldsymbol{x}, t; \boldsymbol{A}) \\ &\quad + g_1\nabla\int_0^t \mathrm{d}\tau \overline{\left[\nabla K(\boldsymbol{R}(t)-\boldsymbol{R}(\tau))\delta(\boldsymbol{x}-\boldsymbol{R}(t))\exp\left(\int_0^t \mathrm{d}\tau\, \boldsymbol{AR}\right)\right]}.\end{aligned}$$

Owing to the source term $\exp(\int \mathrm{d}\tau\, \boldsymbol{AR})$, we may replace $\boldsymbol{R}(\tau)$ in the argument of K by the operator $\delta/\delta\boldsymbol{A}(\tau)$, and obtain

$$\partial_t\tilde{P} = \left[D_0\nabla^2 + \boldsymbol{Ax} + g_1\nabla\int_0^t \mathrm{d}\tau\, \nabla K\left(\boldsymbol{x} - \frac{\delta}{\delta\boldsymbol{A}(\tau)}\right)\right]\tilde{P}.$$

We solve the equation for the corresponding Green function $\tilde{G}(\boldsymbol{x}, t; \boldsymbol{A}) \equiv \theta(t)\tilde{P}(\boldsymbol{x}, t; \boldsymbol{A})$

$$\left[\frac{\partial}{\partial t} - D_0\nabla^2 - \boldsymbol{Ax} - g_1\nabla\int_0^t \mathrm{d}\tau\, \nabla K\left(\boldsymbol{x} - \frac{\delta}{\delta\boldsymbol{A}(\tau)}\right)\right]\tilde{G}(\boldsymbol{x}, t; \boldsymbol{A}) = \delta(t)\delta(\boldsymbol{x})$$

by iterations. At zeroth order this yields

$$\tilde{G}_0 = L^{-1} \qquad L \equiv \partial_t - D_0\nabla^2 - \boldsymbol{xA}.$$

For clarity, we introduce another pair of arguments of the Green functions to keep track of the (so far implicit) dependence on the initial conditions: we denote the Green function $\tilde{G}$ of the problem with the initial condition $\boldsymbol{R}(t') = \boldsymbol{x}'$ by $\tilde{G}(\boldsymbol{x}, t; \boldsymbol{x}', t')$. In these terms $\tilde{G}(x, t) = \tilde{G}(x, t; 0, 0)$, and differentiating the definition of the 'bare' propagator $\tilde{G}_0$,

$$[\partial_t - D_0\nabla^2 - Ax]\tilde{G}_0(x, t; 0, 0) = \delta(t)\delta(x) \qquad (6)$$

with respect to A, we obtain

$$\frac{\delta\tilde{G}_0(x, t; 0, 0)}{\delta A(\tau)} = \int dx'\, \tilde{G}_0(x, t; x', \tau)x'\tilde{G}_0(x', \tau; 0, 0). \qquad (7)$$

The retarded Green functions on the right-hand side of this equation make the derivative $\delta\tilde{G}_0/\delta A$ vanish for $\tau < 0$ and $\tau > t$. Therefore

$$\frac{\delta^2\tilde{G}_0(x, t; 0, 0)}{\delta A(\tau')\delta A(\tau)} = \int dx' \int dx'' \begin{cases} \tilde{G}_0(x, t; x'', \tau')x''\tilde{G}_0(x'', \tau'; x', \tau)x'\tilde{G}_0(x', \tau; 0, 0) & \tau' > \tau \\ \tilde{G}_0(x, t; x', \tau)x'\tilde{G}_0(x', \tau; x', \tau')x''\tilde{G}_0(x'', \tau'; 0, 0) & \tau' < \tau. \end{cases}$$

Since $\tilde{G}_0(x, t; x', t') \to \delta(x - x')$, $t \to t'+$, we arrive at the equation

$$\frac{\delta^2\tilde{G}_0(x, t; 0, 0)}{\delta A(\tau)^2} = \int dx'\, \tilde{G}_0(x, t; x', \tau)x'^2\tilde{G}_0(x', \tau; 0, 0). \qquad (8)$$

Analogously

$$\frac{\delta^n\tilde{G}_0(x, t; x', t)}{\delta A(\tau)^n} = \int dx''\, \tilde{G}_0(x, t; x'', \tau)(x'')^n\tilde{G}_0(x'', \tau; x', t') \qquad n = 1, 2, \ldots. \qquad (9)$$

To find the first-order expression for $\tilde{G}$ we calculate the quantity

$$\int_0^t d\tau\, \nabla K\left(y - \frac{\delta}{\delta A(\tau)}\right)\tilde{G}_0(x, t; 0, 0) = \left(t\nabla K(y) + \int_0^t d\tau\, \nabla \sum_{n=1}^{\infty} \frac{(-\nabla)^n K(y)}{n!} \frac{\delta^n}{\delta A^n}\right)\tilde{G}_0$$

making use of the relations (9) we find it to be

$$t\nabla K(y)\tilde{G}_0(x, t; 0, 0) - \nabla K(y) \int_0^t d\tau \int dx'\, \tilde{G}_0(x, t; x', \tau)\tilde{G}_0(x', \tau, 0, 0)$$
$$+ \int_0^t d\tau \int dx'\, \tilde{G}_0(x, t; x', \tau)\nabla K(y - x')\tilde{G}_0(x', \tau, 0, 0). \qquad (10)$$

From the definition of the bare propagator (6) we obtain the relation

$$L[t\tilde{G}_0(x, t; 0, 0)] = \tilde{G}_0(x, t; 0, 0) + tL\tilde{G}_0(x, t; 0, 0) = \tilde{G}_0(x, t; 0, 0)$$

which yields

$$t\tilde{G}_0(x, t; 0, 0) = \int_0^t d\tau \int dx'\, \tilde{G}_0(x, t; x', \tau)\tilde{G}_0(x', \tau; 0, 0).$$

As a consequence of this equation the first two terms cancel on the right-hand side of (10). Therefore

$$\int_0^t d\tau\, \nabla K\left(y - \frac{\delta}{\delta A(\tau)}\right)\tilde{G}_0(x, t; 0, 0)$$
$$= \int_0^t d\tau \int dx'\, \tilde{G}_0(x, t; x', \tau)\nabla K(y - x')\tilde{G}_0(x', \tau; 0, 0). \qquad (11)$$

This relation shows how the 'interaction operator' $\int d\tau\, \nabla K(x - \delta/\delta A(\tau))$ acts on the bare propagator $\tilde{G}_0$. Let us denote the propagator $\tilde{G}_0$ by a full line with an arrow showing the direction of time, and the function K by a broken line. Then (11) may be expressed in a convenient graphical form depicted in figure 1, where the slash on the broken line corresponds to the derivative in (11). It is an essential property of this operator that it acts on a product of $\tilde{G}_0$ with successive time arguments as a first-order differential operator, for instance ($t > t' > 0$)

$$\begin{aligned}\int_0^t d\tau\, \nabla K\left(y - \frac{\delta}{\delta A(\tau)}\right)[\tilde{G}_0(x, t; x', t')\tilde{G}_0(x', t'; 0, 0)] \\ = \left[\int_{t'}^t d\tau\, \nabla K\left(y - \frac{\delta}{\delta A(\tau)}\right) + \int_0^{t'} d\tau\, \nabla K\left(y - \frac{\delta}{\delta A(\tau)}\right)\right] \\ \times [\tilde{G}_0(x, t; x', t')\tilde{G}_0(x', t'; 0, 0)] \\ = \left[\int_{t'}^t d\tau\, \nabla K\left(y - \frac{\delta}{\delta A(\tau)}\right)\tilde{G}_0(x, t; x', t')\right]\tilde{G}_0(x', t'; 0, 0) \\ + \tilde{G}_0(x, t; x', t')\int_0^{t'} d\tau\, \nabla K\left(y - \frac{\delta}{\delta A(\tau)}\right)\tilde{G}_0(x', t'; 0, 0).\end{aligned} \tag{12}$$

This allows for a simple and straightforward construction of the perturbation expansion for $\tilde{G}$, from which we obtain G setting $A = 0$. According to the relations (11) and (12), at every step of the construction of the graphical representation of the perturbation expansion, the operator $\int d\tau\, \nabla K(x - \delta/\delta A(\tau))$ from each graph of the preceding step creates a sum of graphs, in which one of the full lines of the original graph gives rise to two full lines with a broken line emerging from the junction between them, and ending at the left end of the original graph. Finally, a full line corresponding to the extra factor G_0 of the right-hand side of (11) is attached to the left end of the graphs created by the operator $\int d\tau\, \nabla K(x - \delta/\delta A(\tau))$. For the first terms of the perturbation expansion of G we obtain the graphical representation depicted in figure 2, in which

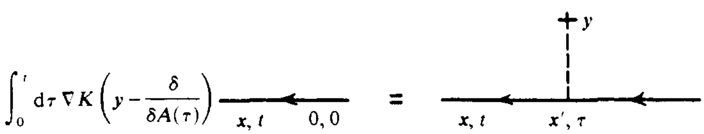


Figure 1. Graphical representation of the action of the operator $\int d\tau\, \nabla K(x - \delta/\delta A(\tau))$ on the bare propagator $\tilde{G}_0$, depicted by a full line with an arrow showing the direction of time. Broken line corresponds to the function K, and the slash to the operator ∇. Integral over the intermediate argument x' is implied.

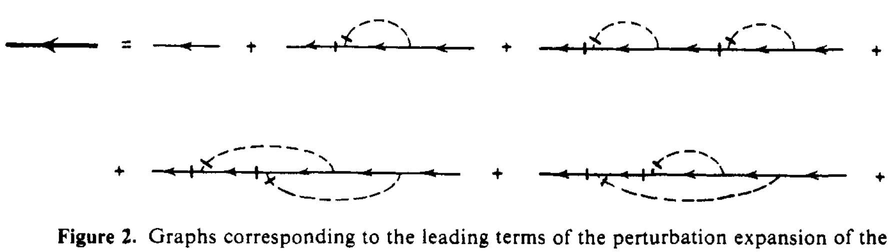


Figure 2. Graphs corresponding to the leading terms of the perturbation expansion of the Green function G of the TSAW. The full lines correspond to the factor $G_0(q, \omega) = 1/(-i\omega + D_0 q^2)$, the broken lines to the factor $K(q) = 1/q^{2\alpha}$, and the slashes to vectors $-iq$, where q is the momentum flowing in the direction shown by the arrow. The coupling constant factor $-g_1$ is prescribed to the three-point vertex with slashes.

the coupling constant factor $-g_1$ is prescribed to the three-point vertex with derivatives There is no frequency flow through the broken lines so far. Obviously, this is the perturbation expansion of the full propagator

$$\begin{aligned} G(x-x', t-t') &= \langle\varphi(x,t)\tilde{\varphi}(x',t')\rangle \\ &= \int \mathrm{D}\varphi\,\mathrm{D}\tilde{\varphi}\,\mathrm{D}\psi\,\mathrm{D}\tilde{\psi}\,\varphi(x,t)\tilde{\varphi}(x',t')\exp(S) \\ &\quad\times\det[\partial_t - D_0\nabla^2 - g_1\nabla(\nabla\psi) - \tilde{\psi}] \end{aligned} \tag{13}$$

where the action S is of the form

$$\begin{aligned} S = -\int \mathrm{d}t\,\mathrm{d}x\,\mathrm{d}x'\,\tilde{\psi}(x,t)K^{-1}(x-x')\partial_t\psi(x',t) + \int \mathrm{d}t\,\mathrm{d}x\{\tilde{\varphi}(x,t)[-\partial_t + D_0\nabla^2]\varphi(x,t) \\ - g_1\nabla\psi(x,t)\varphi(x,t)\nabla\tilde{\varphi}(x,t) + \tilde{\psi}(x,t)\varphi(x,t)\tilde{\varphi}(x,t)\}. \end{aligned} \tag{14}$$

The effect of the determinant in the functional integral (13) is to remove graphs with loops corresponding to $\int \mathrm{d}q\,\mathrm{d}\omega\,G_0(q,\omega)$, which are not present in the original construction of figure 2 (closed loops with two or more retarded propagators G_0 vanish automatically). However, we shall be using dimensional regularisation, in which such terms are zero by definition, and therefore this determinant does not show at all. The auxiliary fields ψ and $\tilde{\psi}$ are time dependent in the action (14) (which corresponds to the propagator $\langle\psi(x,t)\tilde{\psi}(x',t')\rangle = \theta(t-t')K(x-x')$ instead of the factor $K(x-x')$ in the original expansion of figure 2) to reproduce correctly the correlation between the location of derivatives in the expansion of figure 2 and the direction of the time flow: of the two vertices connected by each broken line the vertex with derivatives corresponds to a later moment of time than the vertex without derivatives. Therefore, the graphs of figure 2 when regarded as graphs of the field theory (14), contain loop integrals over frequencies. However, when these frequency integrals are performed, the original 'static' graphs result.

The action (14) covers actually both long-range and short-range versions of the TSAW, but it is convenient to integrate out the axuiliary fields ψ and $\tilde{\psi}$ in the short-range case, which leads to the action

$$\begin{aligned} S = \int \mathrm{d}t\,\mathrm{d}x\,\tilde{\varphi}(x,t)[-\partial_t + D_0\nabla^2]\varphi(x,t) \\ - g_1\int \mathrm{d}t\,\mathrm{d}t'\,\mathrm{d}x\,\varphi(x,t)\nabla\tilde{\varphi}(x,t)\theta(t-t')\nabla[\varphi(x,t')\tilde{\varphi}(x,t')] \end{aligned} \tag{15}$$

with interaction local in space but non-local in time.

3. Renormalisation of the model and calculation of the anomalous dimensions of parameters

The upper critical dimension of the field theory (15) is two, and power counting shows that, in addition to the original interaction vertex, two more four-point interaction terms are at least marginal and have to be included in the interaction, which leads to the interaction $S_{\rm int}$ of the form

$$S_{\rm int}=\int \mathrm{d}t\,\mathrm{d}t'\,\mathrm{d}\boldsymbol{x}\{-g_1\varphi(\boldsymbol{x},t)\nabla\tilde{\varphi}(\boldsymbol{x},t)\theta(t-t')\nabla[\varphi(\boldsymbol{x},t')\tilde{\varphi}(\boldsymbol{x},t')]$$
$$+g_2\nabla\varphi(\boldsymbol{x},t)\nabla\tilde{\varphi}(\boldsymbol{x},t)\theta(t-t')\varphi(\boldsymbol{x},t')\tilde{\varphi}(\boldsymbol{x},t')$$
$$+g_3\varphi(\boldsymbol{x},t)\nabla\tilde{\varphi}(\boldsymbol{x},t)\theta(t-t')\nabla\varphi(\boldsymbol{x},t')\tilde{\varphi}(\boldsymbol{x},t')\} \qquad (16)$$

where the coupling constants corresponding to the new vertex structures are denoted by g_2 and g_3. The RG analysis of the field theory has been carried out at one-loop level [3] but, unfortunately, it is not sufficient to consider only the four-point interaction (16), as was pointed out recently [5]. Consider the one-loop graphs of figure 3, which contribute to the four-point vertex renormalisation (for brevity, we have shown graphs containing the g_1 vertex only). In these graphs, the dotted line corresponds to the θ function factor of the retarded interaction (16). The graph of figure 3(c), for instance, corresponds to the following analytic expression (for simplicity, we set the external momenta flowing in the lower vertices equal to zero):

$$I_{2c}(\boldsymbol{p},\omega_l)=-g_1^2\int\frac{\mathrm{d}\boldsymbol{q}}{(2\pi)^d}\int\frac{\mathrm{d}\omega}{2\pi}\frac{1}{[-\mathrm{i}(\omega+\omega_3)+o][\mathrm{i}(\omega+\Sigma_{l=1}^3\omega_l)+o]}$$
$$\times\frac{(\boldsymbol{q},\boldsymbol{p}+\boldsymbol{q})(\boldsymbol{q},\boldsymbol{p})}{[\mathrm{i}\omega+D_0\boldsymbol{q}^2][-\mathrm{i}(\omega+\omega_1+\omega_3)+D_0(\boldsymbol{p}+\boldsymbol{q})^2]}.$$

Integrating over the frequency we obtain

$$I_{2c}(\boldsymbol{p},\omega_l)=-g_1^2\int\frac{\mathrm{d}\boldsymbol{q}}{(2\pi)^d}(\boldsymbol{q},\boldsymbol{p}+\boldsymbol{q})(\boldsymbol{q},\boldsymbol{p})$$
$$\times\left(\frac{1}{(-\mathrm{i}\omega_3+D_0\boldsymbol{q}^2)(\mathrm{i}\,\Sigma_{l=1}^3\omega_l-D_0\boldsymbol{q}^2)[-\mathrm{i}(\omega_1+\omega_3)+D_0(\boldsymbol{q}+\boldsymbol{p})^2+D_0\boldsymbol{q}^2]}\right.$$
$$\left.+\frac{1}{[\mathrm{i}(\omega_1+\omega_2)+o](-\mathrm{i}\,\Sigma_{l=1}^3\omega_l+D_0\boldsymbol{q}^2)[\mathrm{i}\omega_2+D_0(\boldsymbol{q}+\boldsymbol{p})^2]}\right).$$

The first term here is obviously convergent at large momenta, whereas the momentum integral is divergent in the second term, which contains a factor $1/[\mathrm{i}(\omega_1+\omega_2)+o]$ corresponding to the retarded structure of the interaction vertices in (16). This factorisation is a generic feature of the vertex graphs; therefore the relevant contribution of each graph to the vertex renormalisation constants is given by an effective 'static' graph with momentum integration only, in which the θ function factors of dotted lines are replaced by unity and the internal frequencies are set equal to zero.

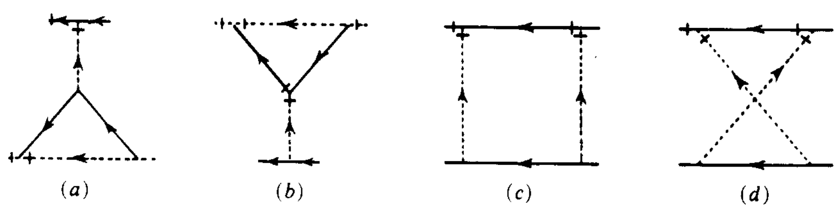


Figure 3. One-loop graphs of the short-range TSAW field theory (15) and (16), which renormalise the four-point interaction term. For brevity, we have depicted graphs with g_1 vertex only. The dotted line corresponds to the factor $1/(-\mathrm{i}\omega+o)$, where ω is the frequency flowing through the line. These graphs give rise also to the vertex with g_2, and for the general case (16) similar graphs with all vertex structures (and thus with differently arranged slashes) must be taken into account.

The reason why it is not sufficient to consider the four-point interaction only is that g_2 vertices can be attached in arbitrary number to these graphs without any change in the dimensionality of the corresponding momentum integral: if the g_2 vertex is added in such a way that the derivatives at the end of the dotted line act on internal lines, then the corresponding momenta dimensionally compensate for the additional propagator appearing in the extended graph, and the large-momentum behaviour of the integrand is thus the same as in the original graph. This is illustrated by the examples of figure 4. In general, the g_2 vertex is inevitably generated by renormalisation, which therefore gives rise to an infinite set of divergences of different types, i.e. the field theory (15) and (16) is not renormalisable.

In the standard renormalisation scheme, we have to add to the action an interaction term with a coupling constant for each new type of divergent graph, after which we obtain a physically useless action with an infinite number of parameters which are, in general, independent of each other. In the present case, all the new divergences are logarithmic, the upper critical dimension $d_c = 2$ is not changed, and the action (16) remains applicable in the perturbation theory above the upper critical dimension. If, by some symmetry or other reason, the resulting infinite set of interaction terms could be presented in the form of a functional of the fields with a finite number of parameters, the model could remain useful. We have not been able to establish such a connection between the new interaction terms in the present model. Physically, the fact that the action (16) is not renormalisable indicates that some contributions neglected in the derivation of the continuum problem (1) and (2) from the discrete definition [1] probably are essential in determining the asymptotic behaviour of the TSAW analytically.

There are, however, two cases, in which the g_2 vertex is not generated by renormalisation, if it is absent in the initial model. First, if we choose $g_1 = g_2 = 0$, $g_3 \neq 0$, then it may readily be checked that this model is self-consistent in the sense that four-point vertex structures corresponding to g_1 and g_2 are absent in the perturbation expansion. Nevertheless, this model also turns out to be non-renormalisable, and the reason is that it contains another set of divergent 1PI graphs with arbitrary number of external φ and $\tilde{\varphi}$ legs. Examples of these are shown in figure 5. Second, if we choose $g_2 = 0$, $g_1 = g_3 \neq 0$, the interaction (16) takes the form

$$S_{\text{int}} = -g_1 \int \mathrm{d}t\, \mathrm{d}t'\, \mathrm{d}\boldsymbol{x}\, \varphi(\boldsymbol{x}, t)\nabla\tilde{\varphi}(\boldsymbol{x}, t)\theta(t - t')\varphi(\boldsymbol{x}, t')\nabla\tilde{\varphi}(\boldsymbol{x}, t') \tag{17}$$

which, apart from the retarded character of the interaction, coincides with the interaction of the field-theoretic model of random walk in random environment ('random

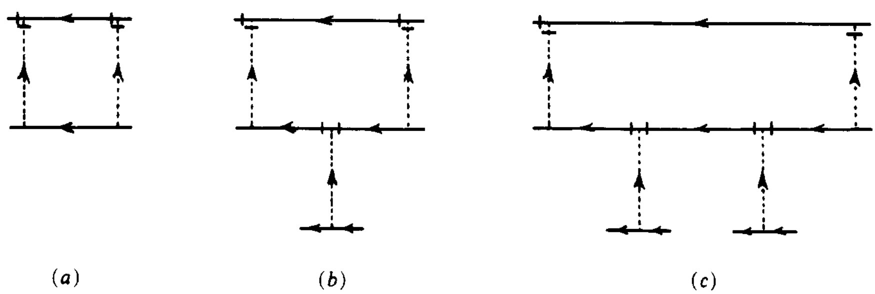
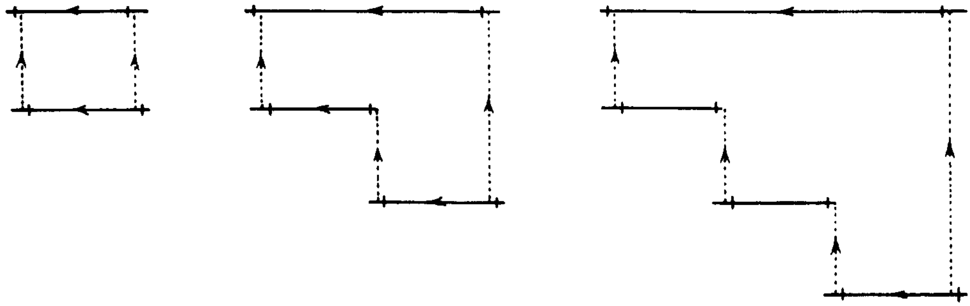


Figure 4. Examples of the structure of divergent one-loop graphs from the 1PI graphs of figure 2: (*a*) is the original four-point graph and (*b*) and (*c*) are, respectively, six and eight point graphs obtained by attaching g_2 vertices to it.



(*a*) (*b*) (*c*)

Figure 5. Examples of divergent one-loop graphs of the short-range model (16) in the case $g_1 = g_2 = 0$, $g_3 \neq 0$. Graphs (*a*), (*b*) and (*c*) yield divergent contributions to four, six- and eight-point 1PI Green functions, respectively, and it is clear from these examples that there are similar contributions to Green functions of arbitrary order.

random walk') [9]. The essential feature of the interaction (17) is that the $\tilde{\varphi}$ field enters only with a derivative. In the graphs of perturbation theory, the corresponding momenta therefore factorise at the vertices with external $\tilde{\varphi}$ legs, thus rendering all graphs with more than two external $\tilde{\varphi}$ legs superficially convergent (i.e. they yield only finite contributions to renormalisation constants after subtraction of divergences corresponding to divergent subgraphs). As a result, the model with the interaction (17) is multiplicatively renormalisable with the same long-time asymptotic behaviour as in the model of random walk in random velocity field [9]. From these arguments it follows that, apart from the special case $g_1 = g_3$, $g_2 = 0$, the previous analyses [3] of this model are not sufficient to determine the correct long-time behaviour for $d \leqslant 2$.

The situation is, however, different in the case of the long-range TSAW. In the formalism of two fields φ and $\tilde{\varphi}$ the kernel (4) would lead to non-local in space interactions, and to avoid them we prefer not integrate out the auxiliary scalar fields ψ, $\tilde{\psi}$ of the action (14), and introduce auxiliary vector fields $\boldsymbol{A}$, $\tilde{\boldsymbol{A}}$ to 'localise' the interaction corresponding to the second and third terms in the right-hand side of (16). For the retarted Green function G of the long-range model we thus write the functional integral in the form

$$G(\boldsymbol{x}-\boldsymbol{x}', t-t') = \int \mathrm{D}\varphi\, \mathrm{D}\tilde{\varphi}\, \mathrm{d}\psi\, \mathrm{D}\tilde{\psi}\, \mathrm{D}\boldsymbol{A}\, \mathrm{D}\boldsymbol{A}\, \mathrm{D}\tilde{\boldsymbol{A}}\, \varphi(\boldsymbol{x}, t)\tilde{\varphi}(\boldsymbol{x}', t') \exp(S)$$

with the action S

$$\begin{aligned} S = -\int & \mathrm{d}t\, \mathrm{d}\boldsymbol{x}\, \mathrm{d}\boldsymbol{x}' [\tilde{\boldsymbol{A}}(\boldsymbol{x}, t) K^{-}(\boldsymbol{x}-\boldsymbol{x}')\partial_t \boldsymbol{A}(\boldsymbol{x}', t) + \tilde{\psi}(\boldsymbol{x}, t) K^{-1}(\boldsymbol{x}-\boldsymbol{x}')\partial_t \psi(\boldsymbol{x}', t)] \\ & + \int \mathrm{d}t\, \mathrm{d}\boldsymbol{x} \{\tilde{\varphi}(\boldsymbol{x}, t)[-\partial_t + D_0\nabla^2]\varphi(\boldsymbol{x}, t) - (g_1 - g_3)\nabla\psi(\boldsymbol{x}, t)\varphi(\boldsymbol{x}, t)\nabla\tilde{\varphi}(\boldsymbol{x}, t) \\ & + g_2\psi(\boldsymbol{x}, t)\nabla\varphi(\boldsymbol{x}, t)\nabla\tilde{\varphi}(\boldsymbol{x}, t) + \tilde{\psi}(\boldsymbol{x}, t)\varphi(\boldsymbol{x}, t)\tilde{\varphi}(\boldsymbol{x}, t) \\ & + g_3\boldsymbol{A}(\boldsymbol{x}, t)\varphi(\boldsymbol{x}, t)\nabla\tilde{\varphi}(\boldsymbol{x}, t) - \tilde{\boldsymbol{A}}(\boldsymbol{x}, t)\varphi(\boldsymbol{x}, t)\nabla\tilde{\varphi}(\boldsymbol{x}, t)\} \end{aligned} \tag{18}$$

where the kernel K is defined by the relation (4), and we have used the same notation for the coupling constants as in the short-range case (16). We have cast the local action in the manifestly multiplicatively renormalisable form (18) instead of the action presented earlier [5]. These expressions are connected by a simple transformation of the auxiliary fields: shifting the fields in the action (18) according to $\boldsymbol{A} \to \boldsymbol{A} - \nabla\psi$, $\tilde{\psi} \to \tilde{\psi} - \nabla \tilde{\boldsymbol{A}}$ and integrating by parts in the last term we recover the previous form of the local action [5]. The field theory for the long-range TSAW problem (2)-(4) corresponds to the choice $g_1 \neq 0$, $g_2 = g_3 = 0$ in the action (18). Power counting shows that the upper critical dimension d_c is now equal to $d_c = 2 + 2\alpha$, and, the most remarkable difference from the short-range case, the 'bad' one-particle irreducible (1PI) four-point graphs of figure 3 are convergent for finite $\alpha > 0$. It is not difficult to see that all 1PI graphs, which have more than two external φ and $\tilde{\varphi}$ legs, do not contain superficial divergences in this case, and therefore they are not relevant in the RG sense. However, three-point graphs with external φ and $\tilde{\varphi}$ legs (examples of which are the triangular 1PI subgraphs in figure 3) still contain, in general, superficial divergences, and they give rise to the renormalisation of the interaction vertices in (18). It should be noted that the structure of each three-point interaction vertex in the action (18) is preserved separately under renormalisation: e.g. if we set $g_1 \neq 0$, and $g_2 = g_3 = 0$, then vertices corresponding to the coupling constants g_2 and g_3 are not generated by renormalisation. To see this in the case $g_1 = g_2 = 0$, $g_3 \neq 0$ it is convenient to cast the effective interaction in the form $g_3 \boldsymbol{A}\varphi\nabla\tilde{\varphi} + \tilde{\boldsymbol{A}}\tilde{\varphi}\nabla\varphi$ for which the graphical analysis is simple. This implies, in particular, that the field-theoretic version of the long-range TSAW [4] (with $g_1 \neq 0$ only) is multiplicatively renormalisable, whereas in the short-range case the interaction corresponding to g_2 appears with the subsequent proliferation of marginal operators. It is not difficult to see, however, that in the presence of the g_2 vertex also the general long-range model (18) ceases to be renormalisable; indeed, if we attach g_2 vertices to logarithmically divergent three-point graphs, this does not change the large-momentum behaviour of the corresponding loop integral and we are again faced with the problem of generation of an infinite set of marginal operators. Therefore, we conclude that even in the long-range case the field theory (18) is renormalisable, if and only if $g_2 = 0$.

Summarising the results of the preceding analysis we arrive at the conclusion that the short-range field theory of the TSAW (16) is renormalisable only if $g_1 = g_3$, $g_2 = 0$, in which case it coincides with the model of random walk in unconstrained random field [9]. The long-range generalisation (18) is renormalisable if $g_2 = 0$, and coincides with the unconstrained long-range random random walk problem [11, 12], when $g_1 = g_3$.

We have carried out the RG analysis of the long-range TSAW ($g_1 \neq 0$, $g_2 = g_3 = 0$) to two-loop order. For convenience, we introduce a new coupling constant $u_{1,0} = g_1 D_0^{-2} C_\alpha$, where $C_\alpha = [2^{1+2\alpha}\pi^{1+\alpha}\Gamma(1+\alpha)]^{-1}$, and write the basic action [13] (which contains renormalised parameters only) in the form

$$S_B = -\int \mathrm{d}t\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}\boldsymbol{x}'\,\tilde{\psi}(\boldsymbol{x}, t) K^{-1}(\boldsymbol{x}-\boldsymbol{x}')\partial_t \psi(\boldsymbol{x}', t) + \int \mathrm{d}t\,\mathrm{d}\boldsymbol{x}\{\tilde{\varphi}(\boldsymbol{x}, t)[-\partial_t + D\nabla^2]\varphi(\boldsymbol{x}, t)$$
$$- u_1\mu^\varepsilon D^2 C_\alpha^{-1} \nabla\psi(\boldsymbol{x}, t)\varphi(\boldsymbol{x}, t)\nabla\tilde{\varphi}(\boldsymbol{x}, t) + \tilde{\psi}(\boldsymbol{x}, t)\varphi(\boldsymbol{x}, t)\tilde{\varphi}(\boldsymbol{x}, t)\} \qquad (19)$$

where μ is the scale setting parameter, and $\varepsilon = 2 + 2\alpha - d$. The renormalised parameters of this action are related to the bare ones as

$$D_0 = Z_D D \qquad u_{1,0} = u_1 \mu^\varepsilon Z_1 Z_D^{-2} \qquad (20)$$

where Z_D and Z_1 are, respectively, the renormalisation constants of the diffusion coefficient, and the first interaction vertex of the action (19) (the second vertex is not renormalised).

The relation between the renormalised (G) and bare ($G^{(0)}$) Green functions

$$G(\omega, \boldsymbol{q}; u_1, D, \mu) = G^{(0)}(\omega, \boldsymbol{q}; u_{1,0}, D_0)$$

yields the equation

$$\left(\mu\frac{\partial}{\partial\mu}+\beta_1\frac{\partial}{\partial u_1}+\gamma_D D\frac{\partial}{\partial D}\right)G(\omega,\boldsymbol{q};u_1,D,\mu)=0 \tag{21}$$

where the beta function β_1 is defined as

$$\beta_1\equiv\mu\frac{\partial}{\partial\mu}\bigg|_0 u_1=u_1(-\varepsilon+\gamma_1-2\gamma_D)$$

and the functions γ as

$$\gamma_x=-\mu\frac{\partial}{\partial\mu}\bigg|_0 \ln Z_x \qquad x=1, D.$$

The subscript '0' indicates that the partial derivatives are taken at fixed values of the bare parameters $u_{1,0}$ and D_0. From (21) and dimensional arguments we obtain for the mean square displacement $\overline{\boldsymbol{R}^2(t)}$ of the random walk the following equation

$$\left((2+\gamma_D)t\frac{\partial}{\partial t}+\beta_1\frac{\partial}{\partial u_1}-2\right)\overline{\boldsymbol{R}^2(t)}=0$$

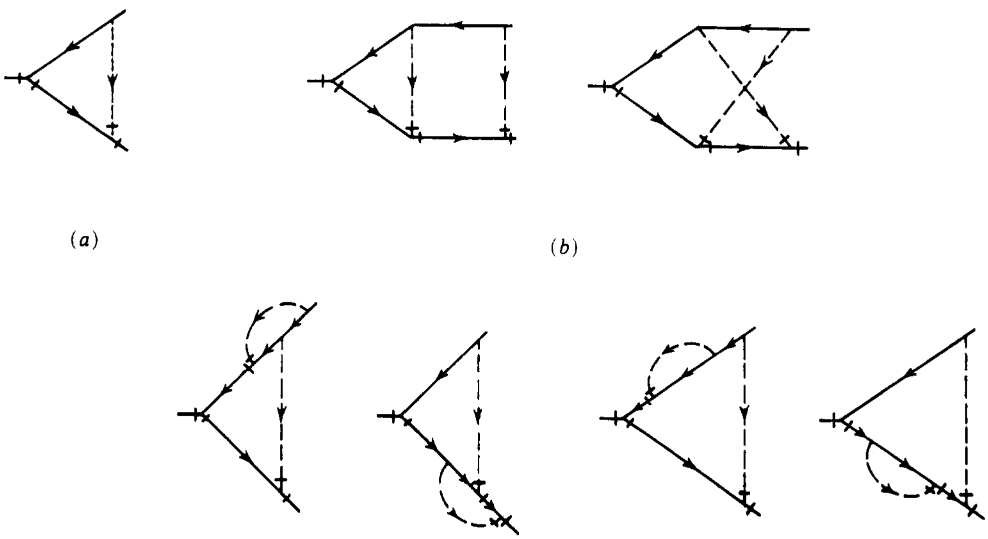
from which the asymptotic behaviour of $\overline{\boldsymbol{R}^2(t)}$ at long times may be extracted.

We use dimensional regularisation with minimal subtractions, and calculating the contribution of the self-energy graphs depicted in figure 2, we obtain for the renormalisation coefficient Z_D of the diffusion coefficient the expression

$$Z_D=1-\frac{u_1}{(1+\alpha)\varepsilon}+\frac{u_1^2}{4(1+\alpha)^2\varepsilon^2}\left(-1+\frac{2+5\alpha+2\alpha^2}{2\alpha(1+\alpha)}\varepsilon\right) \tag{22}$$

whereas the renormalisation constant Z_1 of the interaction vertex is extracted from the vertex graphs of figure 6 in the form

$$Z_1=1-\frac{u_1}{2(1+\alpha)\varepsilon}+\frac{u_1^2}{8(1+\alpha)^2\varepsilon^2}\left(-2+\frac{2+4\alpha+\alpha^2}{\alpha(1+\alpha)}\varepsilon\right). \tag{23}$$



(a) (b)

Figure 6. One-particle irreducible vertex graphs, which contribute to the renormalisation constant Z_1 at one-loop (*a*) and two-loop (*b*) level.

Note the poles at $\alpha=0$ in the second-order terms. They appear due to the fact that two-loop graphs contain the 1PI four-point graphs similar to those of figure 3 as subgraphs, which are convergent for finite α, but become divergent in the limit $\alpha\to 0$. This shows in the relations (22) and (23) in the form of poles $1/\alpha$. This is typical of models with long-range correlations [12, 14], and at some $\alpha>\alpha^*=O(\varepsilon)$ leads to crossover from the scaling behaviour described by the model with long-range correlations to the scaling described by the short-range correlated model. We shall discuss this in detail in the next section.

From (20)-(23) we obtain the following expression for the beta function:

$$\beta_1(u_1)=u_1\left(-\varepsilon+\frac{3}{2(1+\alpha)}u_1-\frac{2+6\alpha+3\alpha^2}{4\alpha(1+\alpha)^3}u_1^2\right). \tag{24}$$

The non-trivial fixed point

$$u_1^*=\tfrac{2}{3}(1+\alpha)\varepsilon\left(1+\frac{2+6\alpha+3\alpha^2}{9\alpha(1+\alpha)}\varepsilon+O(\varepsilon^2)\right) \tag{25}$$

is perturbatively infrared stable, and thus controls the long-distance and long-time behaviour of the model. The anomalous dimension of the diffusion coefficient is the value of the function

$$\gamma_D=\mu\frac{\partial\ln D}{\partial\mu}\bigg|_0 \tag{26}$$

at the fixed point of RG. For $d<d_c=2+2\alpha$, we obtain

$$\gamma_D^*=\gamma_D(u_1=u_1^*)=-\tfrac{2}{3}\varepsilon+\frac{2+3\alpha}{27\alpha(1+\alpha)}\varepsilon^2$$

which determines the exponent ν through the relation $\nu=1/(2+\gamma_D^*)$, and thus the asymptotic behaviour of the model at the long-time limit. For the mean-square displacement of the random walker we obtain

$$\overline{\boldsymbol{R}^2(t)}\sim t^{2\nu}\sim t^{1+\varepsilon/3-(2-3\alpha-6\alpha^2)\varepsilon^2/[54\alpha(1+\alpha)]}$$

which implies superdiffusive behaviour. Logarithmic enhancement of diffusion

$$\overline{\boldsymbol{R}^2(t)}\sim t(\ln t)^{2/3}$$

results at the upper critical dimension $d_c=2+2\alpha$.

Finally, we show that in the other renormalisable case with a single coupling constant ($g_1=g_2=0$, $g_3\neq 0$) of the long-range model (18), the anomalous dimension γ_D^* of the diffusion coefficient may be calculated exactly in the perturbation theory. The reason is that, up to a finite constant, the interaction vertex is not renormalised. The same feature has also been detected in the Navier-Stokes equation with stochastic driving force [15], and in the random random walk problem [12]. In these models 'non-renormalisation' is a consequence of either Galilean invariance or the transverse character of the vector field, whereas in the present case the reason of non-renormalisation is different. Consider the action (18) with the last two interaction terms only, then it is not difficult to see that momenta, which correspond to the derivatives in the interaction terms, factorise in three-point 1PI graphs at the vertices with external φ and $\tilde\varphi$ legs. This renders the remaining loop integrals superficially convergent. In the minimal subtraction scheme the corresponding renormalisation constant is therefore trivial: $Z_3 = 1$. From (26) and the definitions $u_{3,0} = g_3 D_0^{-2} C_\alpha$, $u_{3,0} = u_3 \mu^\varepsilon Z_3 Z_D^{-2}$ we obtain

$$\beta_3(u_3) = \mu \frac{\partial u_3}{\partial \mu}\bigg|_0 = u_3[-\varepsilon - 2\gamma_D(u_3)].$$

From this relation it follows that the fixed-point equation $\beta_3(u_3^*) = 0$ determines the anomalous dimension $\gamma_D^* = \gamma_D(u_3 = u_3^*)$ to all orders in perturbation theory. For the anomalous asymptotic behaviour of the mean-square displacement we obtain

$$\overline{\boldsymbol{R}^2(t)} \sim t^{2/(2-\varepsilon/2)} \qquad d < d_c = 2 + 2\alpha \tag{27}$$

or

$$\overline{\boldsymbol{R}^2(t)} \sim t(\ln t)^{1/2} \qquad d = d_c = 2 + 2\alpha$$

which are the same as in the model of random walk in transverse random velocity field with long-range correlations [12]. It is also interesting to note that the relation (27) corresponds to a value of the exponent ν, which is exactly the same as was predicted in the long-range TSAW problem (2)–(4) from a simple Flory argument [4].

4. Renormalisation of composite operators and crossover from long-range to short-range scaling

We are interested in the anomalous behaviour of the scalar composite operators which correspond to the four-point interaction vertices of the short-range model (16), since they are of importance to investigation of the crossover between the scaling regimes described by long-range and short-range models. For these composite operators, we introduce the notation

$$\begin{aligned} O_1^{(4)} &\equiv \int \mathrm{d}x \int \mathrm{d}t \int \mathrm{d}t' \varphi(x, t) \nabla \tilde{\varphi}(x, t) \theta(t - t') \nabla[\varphi(x, t') \tilde{\varphi}(x, t')] \\ O_2^{(4)} &\equiv \int \mathrm{d}x \int \mathrm{d}t \int \mathrm{d}t' \nabla \varphi(x, t) \nabla \tilde{\varphi}(x, t) \theta(t - t') \varphi(x, t') \tilde{\varphi}(x, t') \\ O_3^{(4)} &\equiv \int \mathrm{d}x \int \mathrm{d}t \int \mathrm{d}t' \varphi(x, t) \nabla \tilde{\varphi}(x, t) \theta(t - t') \nabla \varphi(x, t') \tilde{\varphi}(x, t'). \end{aligned} \tag{28}$$

From the relations between renormalised (Γ) and bare ($\Gamma^{(0)}$) 1PI Green functions (we omit momentum and frequency arguments for brevity)

$$\begin{aligned} \Gamma_{4O_i^{(4)}}(u_1, D, \mu) &\equiv \Gamma_{\tilde{\varphi}\tilde{\varphi}\varphi\varphi O_i^{(4)}}(u_1, D, \mu) \\ &= Z_{ij}^{(4)} \Gamma^{(0)}_{4O_j^{(4)}}(u_{1,0}, D_0) \qquad i = 1, 2, 3 \end{aligned}$$

we obtain

$$\left[\left(\mu \frac{\partial}{\partial \mu} + \beta_1 \frac{\partial}{\partial u_1} + \gamma_D D \frac{\partial}{\partial D}\right) \delta_{ij} + \gamma_{4ij}\right] \Gamma_{4O_j^{(4)}} = 0 \tag{29}$$

where the matrix γ_4 is defined as

$$\gamma_{4ij} = -\mu \frac{\partial}{\partial \mu}\bigg|_0 Z_{il}^{(4)} (Z^{(4)})_{lj}^{-1}.$$

Dimensional analysis yeilds

$$\Gamma_4(\{\boldsymbol{q}_j\},\{\omega_l\};u_1,D,\mu)=D\mu^{2-d}F\left(\left\{\frac{\boldsymbol{q}_j}{\mu}\right\},\left\{\frac{\omega_l}{\mu^2 D}\right\};u_1\right) \tag{30}$$

and

$$\Gamma_{4O_i^{(4)}}(\{\boldsymbol{q}_j\},\{\omega_l\};u_1,D,\mu)=D^{-1}F_i\left(\left\{\frac{\boldsymbol{q}_j}{\mu}\right\},\left\{\frac{\omega_l}{\mu^2 D}\right\};u_1\right). \tag{31}$$

From the relations (29)-(31) we obtain the following expressions for the scaling dimensions of the composite operators $O^{(4)}$

$$d_{4,i}=d-2-2\gamma_D(u_1^*)+\lambda_i^{(4)} \qquad i=1,2,3 \tag{32}$$

where $\lambda^{(4)}$ are the eigenvalues of the matrix γ_4 at the fixed point of the RG equations. It should be noted that, in general, these scaling dimension are not prescribed to the composite operators themselves, but to their certain linear combinations.

Calculating the one-loop graphs of figure 7, we arrive at the matrix

$$\gamma_4=\frac{u_1}{1+\alpha}\begin{pmatrix}\frac{3}{2}+2\alpha & 2 & 0\\ \frac{3}{2} & 1+2\alpha & 0\\ 1 & 1 & 2\alpha\end{pmatrix}$$

which has, at the fixed point of RG, the following eigenvalues:

$$\lambda_1^{(4)}=\tfrac{2}{3}(3+2\alpha)\varepsilon$$

$$\lambda_2^{(4)}=\tfrac{1}{3}(-1+4\alpha)\varepsilon$$

$$\lambda_3^{(4)}=\tfrac{4}{3}\alpha\varepsilon.$$

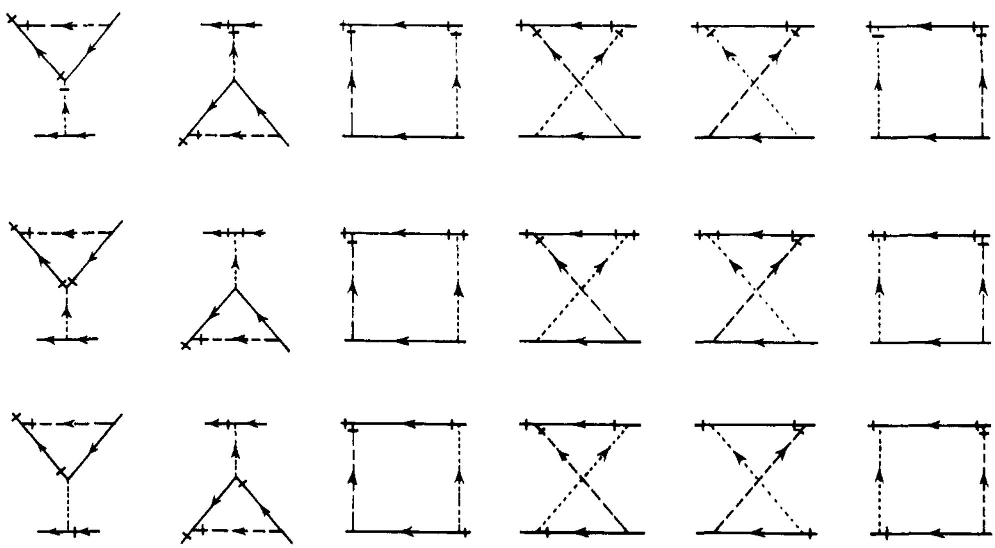


Figure 7. Graphs, which, at one-loop level, renormalise the three composite operators $\int \mathrm{d}x\,\mathrm{d}t\,\mathrm{d}t'\,\varphi\nabla\tilde{\varphi}\theta(t-t')\nabla(\varphi\tilde{\varphi})$, $\int \mathrm{d}x\,\mathrm{d}t\,\mathrm{d}t'\,\nabla\varphi\nabla\tilde{\varphi}\theta(t-t')\varphi\tilde{\varphi}$, and $\int \mathrm{d}x\,\mathrm{d}t\,\mathrm{d}t'\,\varphi\nabla\tilde{\varphi}\theta(t-t')\nabla\varphi\tilde{\varphi}$ corresponding to the four-point interaction terms of the short-range model (16).

At one-loop level, the third eigenvalue corresponds to anomalous scaling of the operator $O_3^{(4)}$, whereas the others are to be prescribed to linear combinations of all three operators (28). Substituting these values to the relations (32) we find the scaling dimensions of these combinations of composite operators, which all turn out to be positive for $\alpha > 0$, $\varepsilon > 0$.

We now turn to the analysis of the effects of the poles at $\alpha = 0$ appearing in the coefficients of the RG equations. These singularities prevent the limit $\alpha \to 0$ in the formulae for the long-range model and thus signal that additional divergences appear in this limit. The crossover between scaling regimes described by pairs of short-range and long-range correlated field theories has been analysed earlier [12, 14], and we proceed in the same fashion as in the analysis of the model of random walk in random environment [12] ('random random walk'), which is closely related to the present model, There, the anomalous dimension of the relevant composite operator of the type (28) turns out to be negative at the fixed point of RG, and therefore for small enough values of α the scaling dimension of the composite operator becomes negative, indicating that the operator has become relevant, and thus it has to be taken into account in the renormalisation procedure. However, all three anomalous dimensions in the present case turn out to be positive for $\alpha > 0$, $\varepsilon > 0$, and therefore do not yield any sign of the scaling description of the long-range model becoming inconsistent at $\alpha \to 0$. This situation is similar to that in the analysis of the crossover between long-range and short-range scaling behaviour in the φ^4 model [14]. There is, however, another source of instability of the long-range scaling: unlike the random random walk model, the singular at $\alpha = 0$ term in the expression for the beta function (24) is negative, and thus for small enough α the fixed point (25) becomes unstable and eventually disappears! Indeed, from (24) and (25) we obtain

$$\left.\frac{\mathrm{d}\beta(u_1)}{\mathrm{d}u_1}\right|_{u_1=u_1^*} = \varepsilon - \frac{2+6\alpha+3\alpha^2}{9\alpha(1+\alpha)}\varepsilon^2 + \mathrm{O}(\varepsilon^3)$$

and in the limit $\alpha \to 0$ this expression changes sign at $\alpha^* = 2\varepsilon/9$, which then can be regarded as the limit of applicability of the RG analysis of the long-range model.

5. Conclusion

We have investigated the asymptotic behaviour of the model of the 'true' self-avoiding random walk for both short-range and long-range repulsion. We have presented a novel short derivation of the corresponding field theory, and analysed its renormalisability in both cases. We have shown that the short-range version is not renormalisable and is thus lacking predictive power, apart from the case $g_1 = g_3$, $g_2 = 0$, which corresponds to the model of random walk in unconstrained random velocity field. On the other hand, the long-range model is shown to be renormalisable on the plane $g_2 = 0$ in the three-dimensional space of interaction parameters g_1, g_2 and g_3. For the case $g_1 > 0$, $g_2 = g_3 = 0$, which corresponds to the original problem of the long-range TSAW, we have calculated the beta function and anomalous dimension of the diffusion coefficient to two-loop order. Above the upper critical dimension $d > d_c = 2 + 2\alpha$ normal diffusion results: $\boldsymbol{R}^2(t) \sim t$, at the upper critical dimension a logarithmic enhancement of diffusion takes place: $\boldsymbol{R}^2(t) \sim t(\ln t)^{2/3}$, whereas at $2 + 2\alpha - d = \varepsilon > 0$ power-like superdiffusive behaviour occurs: $\boldsymbol{R}^2(t) \sim t^{1+\varepsilon/3-(2-3\alpha-6\alpha^2)\varepsilon^2/[54\alpha(1+\alpha)]}$. In the case $g_1 = g_2 = 0$, $g_3 > 0$, we show that the anomalous behaviour of diffusion may be determined perturbatively exactly, leading again to superdiffusive anomaly: $\overline{\boldsymbol{R}^2(t)} \sim t(\ln t)^{1/2}$ at $d = 2+2\alpha$, and $\boldsymbol{R}^2(t) \sim t(\ln t)^{2/(2-\varepsilon/2)}$, when $d < 2+2\alpha$. This result is exactly the same as the Flory argument yields for this model.

We have also calculated the anomalous dimensions of the scalar composite field operators: $\int \mathrm{d}\boldsymbol{x}\,\mathrm{d}t\,\mathrm{d}t'\, \varphi\nabla\tilde{\varphi}\theta(t-t')\nabla(\varphi\tilde{\varphi})$, $\int \mathrm{d}\boldsymbol{x}\,\mathrm{d}t\,\mathrm{d}t'\, \nabla\varphi\nabla\tilde{\varphi}\theta(t-t')\varphi\tilde{\varphi}$, and $\int \mathrm{d}\boldsymbol{x}\,\mathrm{d}t\,\mathrm{d}t'\, \varphi\nabla\tilde{\varphi}\theta(t-t')\nabla\varphi\tilde{\varphi}$ at one-loop accuracy and using the results analysed the crossover from the scaling behaviour described by the long-range model to that described by the short-range one, when $\alpha \to 0$. In this model the non-trivial infrared-stable fixed point becomes unstable at $\alpha \propto \varepsilon$. This is different from the crossover in the closely related model of random walk in random environment, in which the fixed point remains stable for small α, but the four-point operators become relevant, when $\alpha \to 0$ due to the decreasing values of their anomalous dimensions.

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